Solution of Simultaneous Linear Equations

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Outline

- Review last lecture
- Solution of simultaneous equations
- Gauss elimination procedure
- Rules for existence and uniqueness of solutions
- Matrix rank and determinant rank
- Homogenous equations

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Review Vector Spaces

- Vector spaces are a generalization of the rules for physical vectors
- Have simple rules for elements of vector spaces
- Inner product is a generalization of the dot product
- Norm is a generalization of vector length
- Vector space a unifying concept for many of the topics covered in ME 501AB

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Review Norms

q norm definition: $\|\mathbf{x}\|_q = \left[\sum |x_i|^q\right]^{1/q}$

- Norm of vector **x** expressed as ||x|| generalizes notion of vector length
- q norm is one possible norm definition
  - usual vector length is the "two norm", $\|\mathbf{x}\|_2$
  - one norm is sum of absolute values
  - infinity norm is the element with maximum absolute value

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Review Inner Products

- General expression is (**x, y**)
- For two conventional vectors, $[x_1\ \ x_2\ \ x_3\ \ldots\ x_n]$ and $[y_1\ \ y_2\ \ y_3\ \ldots\ y_n]$, the inner product is $\Sigma x_i y_i$
- For two column vectors, **x** and **y**, we can express the inner product as $\mathbf{x}^T\mathbf{y}$
- For two row vectors, **x** and **y**, we can express the inner product as $\mathbf{x}\mathbf{y}^T$
- We can also define inner products as integrals of two functions

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Review Linear (In)dependence

- A set of vectors ***linearly dependent*** if the following equation holds, where at least one of the α_i is not equal to zero.

$$\alpha_1\mathbf{x}_{(1)} + \alpha_2\mathbf{x}_{(2)} + \cdots + \alpha_k\mathbf{x}_{(k)} = \sum_{i=1}^{k}\alpha_i\mathbf{x}_{(i)} = \mathbf{0}$$

- A ***linearly independent*** set of vectors is one that is not linearly dependent.
- Cannot have $\mathbf{x}_{(i)} = \mathbf{0}$ in LI set

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Review n-dimensional space

- An n-dimensional vector space has a set of n linearly independent vectors
- No set of n+1 (or more) linearly independent vectors exist in the space
- Any vector in an n-dimensional space can be represented by a linearly independent combination of n vectors.
- A set of n linearly independent vectors is called a **basis set** and is said to **span the space**

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Review Orthogonal Vectors

- Two vectors whose inner product equals zero are ***orthogonal***.
- A set of n vectors, $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, … , $\mathbf{e}_{(n)}$, are mutually **orthogonal** if $(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = a_i\delta_{ij}$
- For an **orthonormal** set of vectors, $(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = \delta_{ij}$
- The usual unit vectors in mechanics (**i**, **j**, and **k**) are orthonormal

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Review Vector Spaces

- Functions such as sin(nπx/L) form a vector space in the region $0 \le x \le L$.
- The inner product, defined below, shows that this is a set of orthogonal functions

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm}$$

- The set of functions at the right is orthonormal $\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$
- Weight function sometimes used

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Review Simultaneous

- A set of simultaneous linear algebraic equations may have
  - A single (unique) solution
  - No solution
  - An infinite number of solutions
- A linear combination of any two equations can replace one of the equations and not change the solution

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Review Example

- Previous example of N = 3 equations

$$\begin{aligned} 3x + 7y - 3z &= 8 \\ 2x - 4y + z &= -3 \\ 8x + 6y - 2z &= 14 \end{aligned}$$

- As **Ax** = **b**

$$\begin{bmatrix} 3 & 7 & -3 \\ 2 & -4 & 1 \\ 8 & 6 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 14 \end{bmatrix}$$

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Solving **Ax** = **b**

(n x m) (m x 1) (n x 1)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

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Gauss Elimination

- Practical tool for obtaining solutions
- Analytical tool for determining linear dependence or independence
- Basic idea is to manipulate the equations (or data) to make them easier to solve without changing the results
- Systematically create zeros in lower left part of the equations (or data)

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Upper Triangular Form

- Convert original set of equations to

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \cdots & \alpha_{1n-1} & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \cdots & \alpha_{2n-1} & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & \cdots & \cdots & \alpha_{3n-1} & \alpha_{3n} \\ \vdots & \vdots & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{n-1n-1} & \alpha_{n-1n} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

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Gauss Elimination III

- Upper triangular form on previous slide is easily solved by back substitution
- $x_n = \beta_n/\alpha_{nn}$
- $x_{n-1} = (\beta_{n-1} - \alpha_{n-1\,n}x_n)/\alpha_{n-1\,n-1}$, *et cetera*
- General equation for back substitution

$$x_i = \frac{\beta_i - \sum_{j=i+1}^{n} \alpha_{ij} x_j}{\alpha_{ii}} \qquad i = n-1, n-2, \ldots, 1$$

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Gauss Elimination Algorithm

- How do we get the upper triangular form?
- Work on ***augmented matrix***

$$[\mathbf{A}, \mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} & b_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} & b_n \end{bmatrix}$$

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Gauss Elimination Algorithm II

- First step: make all of column 1 below row 1 zero (sets all $a_{r1} = 0$ except a_{11})
  - Row 1 is called pivot row in this step
- Second step: make all of column 2 below row 2 zero (sets all $a_{r2} = 0$ for $r > 2$)
  - Row 2 is called pivot row in this step
- Continue this to produce zeros below the main diagonal, in each column (from k = 3 to n – 1), using all rows from k to n – 1 as the pivot row

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Gauss Elimination Algorithm III

- Steps to the final upper triangular form?
- Change rows 2 to n

$$[\mathbf{A}, \mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \cdots & \alpha_{2m} & \beta_2 \\ 0 & 0 & \alpha_{33} & \cdots & \cdots & \alpha_{3m} & \beta_3 \\ \vdots & \vdots & 0 & \ddots & & \alpha_{4m} & \beta_4 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \alpha_{nm} & \beta_n \end{bmatrix}$$

α and β values differ in each new row

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General Gauss Elimination

- Use each row from row 1 to row n-1 as the "pivot" row
  - Work on each row below the pivot row
    - Multiply pivot row by $a_{row,pivot}/a_{pivot,pivot}$
    - Subtract result from row r (r = pivot+1 to n) to create modified rows where $a_{row,pivot} = 0$
    - Operation requires subtraction for each column of **A** right of pivot column and for **b**
  - Repeat for each row below pivot
- Repeat for rows 1 to n-1 as pivot rows

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Gauss Pseudocode

$$For\ pivot = 1\ to\ n-1$$
$$\left\{ For\ row = pivot + 1\ to\ n \left[For\ column = pivot+1\ to\ n \left(a_{row,column} \leftarrow a_{row,column} - \frac{a_{row,pivot}}{a_{pivot,pivot}} a_{pivot,column} \right) \quad b_{row} \leftarrow b_{row} - \frac{a_{row,pivot}}{a_{pivot,pivot}} b_{pivot} \right] \right\}$$

$\leftarrow$ is called replacement operator

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Solution Details

- Solve the set of equations on the right

$$2x_1 - 4x_2 - 26x_3 = -34\ (i)$$
$$-3x_1 + 2x_2 + 9x_3 = 13\ (ii)$$
$$7x_1 + 3x_2 + 8x_3 = 14\ (iii)$$

- Subtract –3/2 times (i) from equation (ii) and 7/2 times (i) from (iii)

$$\left[-3 - \left(\frac{-3}{2}\right)2\right]x_1 + \left[2 - \left(\frac{-3}{2}\right)(-4)\right]x_2 + \left[9 - \left(\frac{-3}{2}\right)(-26)\right]x_3 = \left[13 - \left(\frac{-3}{2}\right)(-34)\right]$$

$$\left[7 - \left(\frac{7}{2}\right)2\right]x_1 + \left[3 - \left(\frac{7}{2}\right)(-4)\right]x_2 + \left[8 - \left(\frac{7}{2}\right)(-26)\right]x_3 = \left[14 - \left(\frac{7}{2}\right)(-34)\right]$$

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More Details

- Result from first set of operations

$$2x_1 - 4x_2 - 26x_3 = -34$$
$$0x_1 - 4x_2 - 30x_3 = -38$$
$$0x_1 + 17x_2 + 99x_3 = 133$$

- Subtract 17/(-4) times (ii) from (iii)

$$\left[17 - \left(\frac{17}{-4}\right)(-4)\right]x_2 + \left[99 - \left(\frac{17}{-4}\right)(-30)\right]x_3 = \left[133 - \left(\frac{17}{-4}\right)(-38)\right]$$

- Final upper-triangular form

$$2x_1 - 4x_2 - 26x_3 = -34$$
$$0x_1 - 4x_2 - 30x_3 = -38$$
$$0x_1 + 0x_2 - \frac{57}{2}x_3 = -\frac{57}{2}$$

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Back Substitution

- Final upper-triangular form

$$2x_1 - 4x_2 - 26x_3 = -34$$
$$0x_1 - 4x_2 - 30x_3 = -38$$
$$0x_1 + 0x_2 - \frac{57}{2}x_3 = -\frac{57}{2}$$

- Third equation gives $x_3 = 1$
- Second equation $x_2 = \frac{[-38-(-30)x_3]}{-4} = [-38 + 30(1)]/(-4) = 2$
- First equation $x_1 = \frac{[-34-(-26)x_3-(-4)x_2]}{2} = \frac{[-34-(-26)(1)-(-4)(2)]}{2} = \frac{-34+26+8}{2} = 0$

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Do we have a solution to **Ax** = **b**?

- Answer to question based on the rank which is defined as the number of linearly independent rows or columns
- Both definitions are the same (see proof in Kreyszig)
- Use Gauss elimination to determine rank
  - Convert matrix to upper-triangular (row-echelon) form
  - In this form, rank is number of rows with non-zero coefficients

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What is a row-echelon form?

- Apply Gauss elimination to get
  - All zeros below row one in column one
  - All zeros below row two in column two
  - Keep this up until you get to the final row or until there are no more rows with nonzeros
- Count number of rows that are ***not*** all zeros; this is the rank
- This is way to determine linear independence of a set of vectors

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What is the rank of each matrix?

$$\begin{bmatrix} 6 & 0 & 2 & 0 & 0 & 0 & 1 & 6 \\ 0 & 1 & 7 & 8 & 6 & 2 & 8 & 4 \\ 0 & 0 & 2 & 0 & 3 & 5 & 8 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 & 6 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & 6 & 2 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 9 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Finding Rank

- What is rank of matrix, **A**?
- What is maximum possible value for rank? 2
- Lower right matrix is result of applying Gauss elimination to **A**
- What is the rank of **A**?
- Rank **A** = 1 Only 1 non-zero row

$$\mathbf{A} = \begin{bmatrix} 1 & -12 \\ -6 & 72 \\ 13 & -156 \\ -7 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -12 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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Solutions to Ax = b

- For a system of n unknowns
- If rank **A** = rank [**A b**] = n there is a unique solution
- If rank **A** = rank [**A b**] < n there are an infinite number of solutions
- If rank **A** ≠ rank [**A b**] there are no solutions

Memorize this!

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Three Examples

$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $7x_1 + 3x_2 + 8x_3 = -13$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $50.5x_3 = 50.5$	$x_1 = 0$ $x_2 = -7$ $x_3 = 1$
$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $-2x_1 + 10x_2 + 61x_3 = -9$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $0 = 0$	$x_1 = 12 - 8\alpha$ $x_2 = -2.5 - 4.5\alpha$ $x_3 = \alpha$
$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $-2x_1 + 10x_2 + 61x_3 = -8$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $0 = 1$	No solution

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First Example Rank

Original

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 7 & 3 & 8 \end{bmatrix} \qquad [\mathbf{A}\,\mathbf{b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 7 & 3 & 8 & -13 \end{bmatrix}$$

Row-echelon form

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 50.5 \end{bmatrix} \qquad [\mathbf{A}\,\mathbf{b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 50.5 & 50.5 \end{bmatrix}$$

Here we see that rank **A** = rank [**A b**] = number of unknowns = 3 so we have a unique solution

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Second Example Rank

Original
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ -2 & 10 & 61 \end{bmatrix} \quad [\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ -2 & 10 & 61 & -9 \end{bmatrix}$$

Row-echelon form
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank **A** = rank [**A b**] = 2 which is less than the number of unknowns (3) so we have an infinite number of solutions

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Third Example Rank

Original
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ -2 & 10 & 61 \end{bmatrix} \quad [\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ -2 & 10 & 61 & -8 \end{bmatrix}$$

Row-echelon form
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, rank **A** = 2 ≠ rank [**A b**] = 3 therefore we have ***no*** solutions

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Homogenous Equations

- If **b** = **0**, *i.e.*, each b_i = 0, we automatically have rank **A** = rank[**A b**] so we have a solution
- If this rank equals the number of unknowns, we have a unique solution, **x** = **0**
- If this rank is less than the number of unknowns we have an infinite number of solutions

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Homogenous Equation Example

Equations a_{11} = -1
$$-x_1 - 4x_2 + 3x_3 = 0$$
$$-4x_1 + 11x_2 - 6x_3 = 0$$
$$x_1 - 8x_2 + 5x_3 = 0$$

A matrix
$$\mathbf{A} = \begin{bmatrix} -1 & -4 & 3 \\ -4 & 11 & -6 \\ 1 & -8 & 5 \end{bmatrix}$$

Original [**A b**] matrix
$$[\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} -1 & -4 & 3 & 0 \\ -4 & 11 & -6 & 0 \\ 1 & -8 & 5 & 0 \end{bmatrix}$$

Row-echelon form
$$[\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} -1 & -4 & 3 & 0 \\ 0 & 27 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank **A** = rank [**A b**] = 2 < unknowns = 3 so there are infinite solutions

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Homogenous Equation Example II

Equations a_{11} = +1
$$x_1 - 4x_2 + 3x_3 = 0$$
$$-4x_1 + 11x_2 - 6x_3 = 0$$
$$x_1 - 8x_2 + 5x_3 = 0$$

A matrix
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 3 \\ -4 & 11 & -6 \\ 1 & -8 & 5 \end{bmatrix}$$

Original [**A b**] matrix
$$[\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 1 & -4 & 3 & 0 \\ -4 & 11 & -6 & 0 \\ 1 & -8 & 5 & 0 \end{bmatrix}$$

Row-echelon form
$$[\mathbf{A}\ \mathbf{b}] = \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 2.8 & 0 \end{bmatrix}$$

Rank **A** = rank [**A b**] = unknowns = 3 so there is a unique solution (**x** = **0**)

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Rank and Determinants

- Determinant rank, like matrix rank, is the number of linearly independent rows or columns.
- Two equivalent statements: a determinant is zero if
  - its rows are linearly dependent
  - the size of a determinant is larger than its rank

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Practical Determinant Evaluation

- Use Gauss elimination and find the product of the elements on the diagonal
  - A determinant does not change if one row is replaced by a linear combination of that row with another row
  - Gauss elimination converts a determinant into upper-triangular form without changing its value
  - The determinant of an upper-triangular array is the product of the components on the principal diagonal (example next slide)

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Upper Triangular Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{55} \end{vmatrix}$$

$$= a_{11}(-1)^{1+1} a_{22} \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ 0 & a_{44} & a_{45} \\ 0 & 0 & a_{55} \end{vmatrix} = a_{11}a_{22}(-1)^{1+1} a_{33} \begin{vmatrix} a_{44} & a_{45} \\ 0 & a_{55} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33}a_{44}a_{55}$$

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WARNING!!!

- In computer applications of Gauss elimination rows are swapped to reduce numerical error
- When two rows are swapped in a determinant, the sign of the determinant changes
- Determinant calculations with swapped rows count the number of swaps and multiply the result by $(-1)^{\text{NumberOfSwaps}}$

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Homogenous Infinite Solutions

$a_{11} = -1$ Example

$$-x_1 - 4x_2 + 3x_3 = 0$$
$$-4x_1 + 11x_2 - 6x_3 = 0$$
$$x_1 - 8x_2 + 5x_3 = 0$$

A matrix

$$\mathbf{A} = \begin{bmatrix} -1 & -4 & 3 \\ -4 & 11 & -6 \\ 1 & -8 & 5 \end{bmatrix}$$

Row-echelon form

$$\tilde{\mathbf{A}} = \begin{bmatrix} -1 & -4 & 3 \\ 0 & 27 & -18 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Det\ \tilde{\mathbf{A}} = Det\ \mathbf{A} = (-1)(11)(5) + (-4)(-8)(3) + (1)(-4)(-6) - (1)(11)(3) - (-4)(-4)(5) - (-1)(-8)(-6) = -55 + 96 + 24 - 33 - 80 + 48 = 0$$

Det **A** = 0 => solution of $\mathbf{x} \neq \mathbf{0}$ may exist

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Homogenous Infinite Solutions

$a_{11} = +1$ Example

$$x_1 - 4x_2 + 3x_3 = 0$$
$$-4x_1 + 11x_2 - 6x_3 = 0$$
$$x_1 - 8x_2 + 5x_3 = 0$$

A matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 3 \\ -4 & 11 & -6 \\ 1 & -8 & 5 \end{bmatrix}$$

$$Det\ \mathbf{A} = (1)(11)(5) + (-4)(-8)(3) + (1)(-4)(-6) - (1)(11)(3) - (-4)(-4)(5) - (1)(-8)(-6) = 55 + 96 + 24 - 33 - 80 - 48 = 14$$

Row-echelon form

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & -4 & 3 \\ 0 & -5 & 6 \\ 0 & 0 & -2.8 \end{bmatrix}$$

$$Det\ \tilde{\mathbf{A}} = (1)(-5)(-2.8) = 14$$

Det **A** $\neq$ 0 => **x** = **0** because **b** = **0**

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Cramer's Rule

- Can use for small systems of equations

$$x_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \Big/ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad x_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \Big/ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

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Rank and Inverses

- Finding $\mathbf{A}^{-1}$ for an n x n matrix requires the solution of $\mathbf{Ax} = \mathbf{b}$ n times, where $\mathbf{b}$ is one column of the unit matrix
- We cannot solve this equation unless rank $\mathbf{A}$ = n
- An n x n square matrix $\mathbf{A}$ will not have an inverse unless its rank equals its size

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Rank and Inverses II

- We have previously seen the general result for the elements, b_{ij}, of $\mathbf{B} = \mathbf{A}^{-1}$
- $b_{ij} = C_{ji}/\text{Det}(\mathbf{A})$, where C_{ij} is the cofactor of a_{ij}
- We see that b_{ij} is not defined if Det $\mathbf{A}$ = 0
- $\mathbf{A}^{-1}$ does not exist if Det $\mathbf{A}$ = 0
- Det $\mathbf{A}$ = 0 for an n x n determinant shows that Rank $\mathbf{A}$ < n
- Det $\mathbf{A} \neq 0$ and rank $\mathbf{A}$ = n: two equivalent conditions for $\mathbf{A}_{(n \times n)}$ to have an inverse

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